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NON-STATIONARY INFINITE SERVER MODELS AND THEIR RELATIVES.(U)

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Monterey, California

NON-STATIONARY INFINITE SERVER MODELS

AND THEIR RELATIVES

by

D. P. Gaver

and

J. P. LEHOCZKY

October 1978

Approved for public release; distribution unlimited.

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A backward equation technique is used to derive properties of a time-dependent infinite server system subject to compound Poisson demand. The method is used to suggest a model leading to Zipf's law.

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NON-STATIONARY INFINITE SERVER MODELS AND THEIR RELATIVES

by

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1. Introduction

The waiting-time model characterized by (i) Poisson arrivals to (ii) an unlimited number of servers, these characterized by (iii) independent service times of arbitrary distribution--usually called the $M/G/\infty$ system--has a special significance and utility that stems from the simplicity of its solution. That is, if $N(t)$, $t \geq 0$, denotes the number of arrivals being served at time t, conveniently referred to as <u>system occupancy</u>, and if $N(0) = 0$, then $N(t)$ itself has the Poisson distribution. This fact is well-known when the Poisson arrival rate is a constant, λ, and $F(x)$ is the distribution of service times perhaps with finite mean $E[S] = \mu^{-1} < \infty$; in this latter case the limiting distribution $(t \to \infty)$ is always Poisson with parameter $E[S] = \lambda/\mu$; see Parzen (1962). Such a model approximately characterizes

a) the number of occupied channels in a system of parallel, lightly-loaded telephone or communication channels (see Feller (1967)).
b) the number of items undergoing repair in a logistics system,
c) the number of vehicles simultaneously using a city's freeway system (see Newell (1966)).
d) the number of drug "particles" inhabiting a particular organ (compartment) in a human or animal body at time t (see Gaver and Lehoczky (1977)).

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In this paper it is shown that $N(t)$ possesses a Poisson distribution even if the arrival rate is time dependent, denoted by $\lambda(t)$, and if the distribution of the service time S_t, of an arrival at t is itself dependent upon t, i.e. is $F(x;t)$, t being a parameter. Furthermore, if arrivals occur in bunches--is compound Poisson--then $N(t)$ itself has the compound Poisson distribution. Incidentally, if $\lambda(t) = \lambda$, a constant, arriving bunches are geometrically distributed, and service times are exponential, the limiting distribution of $N(t)$ is shown to be the negative binomial. The simple methods used are extended to study multivariate process models, and also to shot-noise, storage, and Zipf's law.

2. A Backward Equation for System Occupancy: Poisson Arrivals

The distribution of $N(t)$ may be approached as follows. Let $N(t,u)$ denote the system occupancy at t that arrived after time u, $0 \leq u \leq t$; clearly one takes $N(t,0) = N(t)$, and $N(t,t) = 0$ with probability one. Let

$$p_n(t,u) = P\{N(t,u) = n\} \tag{2.1}$$

Hold t fixed, and consider the possible events that may occur in a time period of length du. Write $\bar{F}(x,t) = P\{S_t > x\} = 1 - F(x,t)$, and observe that either (a) no arrival occurs that remains a system occupant by time t, an event of probability $1 - \lambda(u)du\,\bar{F}(t-u,u) + o(du)$, or (b) one arrival occurs and

remains an occupant at t, an event of probability $\lambda(u)du\,\bar{F}(t-u,u) + o(du)$. All other possibilities are negligible. Thus

$$p_n(t,u + du) = [1 - \lambda(u)du\,\bar{F}(t-u,u)]\,p_n(t,u) + \lambda(u)du\,\bar{F}(t-u,u)\,p_{n-1}(t,u) + o(du) \tag{2.2}$$

Now subtract $p_n(t,u)$ from each side, divide by du, and let $du \to 0$. The result is the backward equation

$$\frac{-dp_n(t,u)}{du} = -\lambda(u)\,\bar{F}(t-u,u)\,p_n(t,u) + \lambda(u)\,\bar{F}(t-u,u)\,p_{n-1}(t,u), \quad n = 1,2,3,... \tag{2.3}$$

and

$$\frac{-dp_0(t,u)}{du} = -\lambda(u)\,\bar{F}(t-u,u)\,p_0(t,u) \tag{2.4}$$

The equation (2.4) may be integrated from u to t to give

$$p_0(t,u) = \exp[-\int_u^t \lambda(v)\,\bar{F}(t-v)dv] , \tag{2.5}$$

and when $u \to 0$ this shows that

$$p_0(t,0) = P\{N(t)=0\} = \exp[-\int_0^t \lambda(v)\,\bar{F}(t-v,v)dv] .$$

Invocation of the earlier equation (2.3) shows inductively that

$$p_n(t,0) = P\{N(t) = n\}$$

$$= \exp[-\int_0^t \lambda(v)\ \bar{F}(t-v,v)dv]\ \frac{1}{n!}\ [-\int_0^t \lambda(v)\ \bar{F}(t-v,v)dv]^n \tag{2.6}$$

and thus $N(t)$ has a Poisson distribution.

An approach via generating functions is easy and direct. Define

$$E[z^{N(t,u)}] = g(z,t;u) = \sum_{n=0}^{\infty} z^n\ P\{N(t,u) = n\}. \tag{2.7}$$

Note that the contribution to occupancy at time t from arrivals *in* $(u, u + du)$, $\Delta N(t,u) = N(t,u) - N(t, u + du)$ is independent of the contribution from arrivals after $u + du$ by Poisson process properties, so the convolution property of the generating function leads to writing

$$g(z,t;u)$$

$$= [z\ \lambda(u)du\ \bar{F}(t-u,u) + 1 - \lambda(u)du\ \bar{F}(t-u,u)]\ g(z,t;\ u + du) + o(du). \tag{2.8}$$

Next subtract $g(z,t,u + du)$ from each side, divide by du and let $du \to 0$. The result is

$$\frac{dg(z,t;u)}{du} = (z-1)\ \lambda(u)du\ \bar{F}(t-u,\ u)\ g(z,t;u) \tag{2.9}$$

This equation is immediately solved to produce

$$g(z,t;u) = \exp[(z-1) \int_u^t \lambda(v)\ \bar{F}(t-v,v)\ dv], \tag{2.10}$$

which is recognized to be the generating function of the Poisson distribution (2.6).

3. Compound Poisson (Bunched) Arrivals

Next consider the situation in which the number of arrivals that occur together is random. In other words, bunches of arrivals occur together, and the bunch sizes are discrete random variables, so the arrival process is a (time-dependent) compound Poisson; see Feller (1967). Let $B(u)$ be the size of a bunch that arrives at time u; its distribution and generating function are, respectively,

$$\begin{aligned} P\{B(u) = k\} &= b_k(u) \qquad k = 1,2,\ldots \\ E[z^{B(u)}] &= h(z,u) = \sum_{k=1}^{\infty} z^k b_k(u) \end{aligned} \tag{3.1}$$

Again, let $g(z,t,u)$ be the generating function of $N(t,u)$ as in (2.7). Notice that $g(z,t,u)$ represents the contribution to $N(t,u)$ arising from arrivals in $(u, u + du)$ plus the independent contribution from $(u + du, t)$; see (2.8). To derive the g.f. of the contribution from (u, du), denoted by

ΔN, condition on the event of an arrival and bunch size $B(u)$; by independence of the services,

$$E[z^{\Delta N} | \text{arrival in } (u,du), B(u)]$$
$$= [z\bar{F}(t-u,u) + F(t-u,u)]^{B(u)} \tag{3.2}$$

Now removal of the condition on $B(u)$ yields

$$E[z^{\Delta N} | \text{arrival in } (u, u+du)] = h[(z-1)\ \bar{F}(t-u,u) + 1,u]. \tag{3.3}$$

Of course

$$E[z^{\Delta N} | \text{no arrival in } (u, u+du)] = z^0 = 1,$$

so

$$E[z^{\ N}] = \lambda(u)du\ h[(z-1)\ \bar{F}(t-u,u) + 1,u] + 1 - \lambda(u)du + o(du) \tag{3.4}$$

and thus

$$g(z,t,u) = [\lambda(u)du\ h[(z-1)\ \bar{F}(t-u,u) + 1,u] + 1 - \lambda(u)du]\ g(z,t,u+du)$$
$$+ o(du)\ , \tag{3.5}$$

which leads to the differential equation

$$-\frac{dg}{du} = \lambda(u)\ \{h[(z-1)\ \bar{F}(t-u,u) + 1,u] - 1\}\ g(z,t,u), \tag{3.6}$$

The solution of which is the generating function

$$g(z,t;u) = \exp[-\int_u^t \lambda(v)\ \{1-h[(z-1)\ \bar{F}(t-v,v) + 1,\ v]\}dv] \tag{3.7}$$

Although inversion of (3.7) to produce a simple and familiar expression for the probabilities $\{p_n(t),\ n = 0,1,2,\ldots\}$ seems beyond the realm of possibility, certain facts do emerge. For example, differentiation of $g(z,t;0)$ yields moments,

$$E[N(t)] = \left.\frac{dg}{dz}\right|_{z=1} = \int_0^t \lambda(v)\ E[B(v)]\ \bar{F}(t-v,v)dv\ ,$$

$$\mathrm{Var}[N(t)] = E[N(t)] + \int_0^t \lambda(v)\ E[B(v)\cdot(B(v)-1)]\,(\bar{F}(t-v,v))^2 dv, \tag{3.8}$$

and so forth. Note that $\mathrm{Var}[N(t)]/E[T] > 1$ if bunch sizes are sometimes greater than unity, as is to be expected. Furthermore, setting $z = 0$ yields

$$P\{N(t) = 0\} = p_0(t,0) = \exp\{-\int_0^t \lambda(v)\ h[F(t-v,v),v\ ]dv\} \tag{3.9}$$

The form of the general coefficient of z^j may be deduced from consideration of (3.2) and (3.3). The coefficient of z^j in $h[(z-1)\ \bar{F}(t-u,u) + 1,u]$ is the probability of exactly j "successes"--meaning survivals to time t from u in $B(u)$ Bernoulli trials, hence

$$h[(z-1)\,\bar{F}(t-u,u) + 1,\ u)] = \sum_{j=0}^{\infty} z^j c_j(u) \tag{3.10}$$

$$c_j(u) = \sum_{k=0}^{\infty} b_k(u) \binom{k}{j} (\bar{F}(t-u,u))^j\ (F(t-u,u))^{k-j},\ j = 0,1,2,\ldots$$

Thus an alternative expression for g is

$$g(z,t;u) = \exp[-\int_u^t \lambda(v)\ \{1 - \sum_{j=0}^{\infty} z^j c_j(v)\}dv$$

Since $\{c_j(v),\ j = 0,1,2,\ldots\}$ is a discrete probability distribution for every v, as is evident from (3.10), $N(t,u)$ clearly has a compound Poisson distribution.

Of interest is the following

<u>EXAMPLE</u>. Suppose $\lambda(v) = \lambda$, $\bar{F}(x,v) = e^{-\theta x}$, $b_k(u) = (1-\alpha)\alpha^{k-1}$, $k = 1,2,\ldots$. This is the case of stationary "stuttering Poisson" arrivals. Substitution into (3.7) yields for $u = 0$,

$$g(z,t;0) = \exp\left[-\int_0^t \lambda \left\{1 - \frac{(1-\alpha)[(z-1)e^{-\theta(t-v)}+1]}{1 - \alpha[(z-1)e^{-\theta(t-v)}+1]}\right\} dv\right]$$

$$= \exp\left[\lambda\int_0^t \frac{(z-1)\ e^{-\theta(t-v)}dv}{(1 - \alpha - \alpha[(z-1)e^{-\theta(t-v)}]}\right]$$

$$= \exp\left[\frac{-\lambda}{\theta\alpha}\ \ln \left\{\frac{1 - \alpha - \alpha[(z-1)]}{1 - \alpha - \alpha(z-1)e^{-\theta t}}\right\}\right]$$

$$= \left[\frac{1 - \alpha - \alpha(z-1)e^{-\ t}}{1 - \alpha z}\right]^{\lambda/\theta\alpha}$$

as t tends to infinity the latter generating function approaches

$$g(z,\infty;0) = \lim_{t \to \infty} g(z,t;0) = \left(\frac{1-\alpha}{1-\alpha z}\right)^{\lambda/\theta\alpha}$$

the generating function of the negative binomial distribution. Hence the long-run distribution of server occupancy is, in this particular case, a familiar form that may readily be used in various applications in place of the ordinary Poisson that results from (2.6) or (2.10) under similar circumstances.

4. Bunch Division into Two Classes

Suppose that bunches arrive in a Poisson manner, but that each bunch is independently fragmented into subbunches of type 1 and type 2 customers with probability p_1 and p_2 respectively (although these probabilities may be time-dependent also, we do not bother with this). Items of type i are served in accordance with distribution $F_i(x,u)$. This setup may model demands on certain logistics systems, e.g. by landing aircraft with different failure categories. Although the number of types of arrivals is limited to two, there is no difficulty in extending it to more types if necessary.

Following the pattern leading to (3.5) one may write

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$$E[z_1^{N_1(t,u)} z_2^{N_2(t,u)}] = g(z_1,z_2,t;u)$$

$$= \{\lambda(u)du\ h[p_1(z_1-1)\ \bar{F}_1(t-u,u) + p_2(z_2-1)\ \bar{F}_2(t-u,u)+1,u]+1-\lambda(u)du\}$$

$$\times\ g(z_1,z_2,t;u + du) \tag{4.1}$$

where $N_i(t,u)$ is the contribution to system occupancy of the type i arrivals between u and t. It then follows that

$$-\frac{dg}{du} = -\lambda(u)\{1 - h[p_1(z_1-1)\ \bar{F}_1(t-u,u) + p_2(z_2-1)\ \bar{F}_2(t-u,u) + 1,u]$$

$$\times\ g(z_1,z_2,t;u)\}\ , \tag{4.2}$$

from which the joint generating function

$$g(z_1,z_2,t;0)$$

$$= \exp[-\int_0^t \lambda(v)\{1-h[p_1(z_1-1)\ \bar{F}_1(t-v,v) + p_2(z_2-1)\ \bar{F}_2(t-v,v) + 1,v]\}dv \tag{4.3}$$

appears. Moments are obtained by differentiation; see (3.8); to obtain means and variances simply replace $\lambda(u)$ by $p_i\lambda(u)$. The covariance results from partial differentiations of the exponent at $z_1 = 1,\ z_2 = 1$:

$$\mathrm{cov}[N_1(t),N_2(t)] = \int_0^t \lambda(v)\ E[B(v)(B(v)-1)]\ p_1p_2\ \bar{F}_1(t-v,v)\ \bar{F}_2(t-v,v)dv \tag{4.4}$$

which is always positive.

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EXAMPLE. Let $\lambda(u) = \lambda$, $h(z,u) = (1-\alpha)z/(1-\alpha z)$. If $\bar{F}_i(x,u) = e^{-\theta_i x}$ then the following results:

$$g(z_1,z_2,t;0) = \exp\left\{ -\lambda \int_0^t \left\{ \frac{p(z_1-1)e^{-\theta_1 v} + p_2(z_2-1)\, e^{-\theta_2 v}}{1-\alpha+\alpha[p_1(z_1-1)e^{-\theta_1 v}+p_2(z_2-1)e^{-\theta_2 v}]} \right\} dv \right\} \tag{4.5}$$

which, regrettably, cannot be integrated in closed form <u>unless</u> $\theta_1 = \theta_2$ (= 1 for convenience) in that case we find that as $t \to \infty$

$$g(z_1,z_2,\infty,0) = \left[\frac{1-\alpha}{1-\alpha p_1 z_1 - \alpha p_2 z_2}\right]^{\lambda/\alpha} \tag{4.6}$$

This is the generating function of a bivariate distribution with negative binomial marginal distribution.

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5. Related Problems, or "Sons and Daughters of $M/G/\infty$"

Models for shot-noise, see Rice (1954), and for dams and rainfall and runoff, see Gaver and Miller (1962), share the general structure of the previous infinite server models. Time-dependent versions of these will be formulated and briefly discussed using the backward equation approach.

Let $\lambda(u)$ be the rate of arrival of a certain event at time u, and let $\underset{\sim}{e}(u,t)$ denote the random effect at time t of an event at time u, $0 \leq u \leq t$. In general $\underset{\sim}{e}(u,t)$ will be real-valued random variable; in the shot noise application it represents the response at time t of an electrical circuit to an impuse at time u, and in the case of a dam or storage system it may be the amount of water in the reservoir resulting from a rainstorm at time u. Let the Laplace transform of $\underset{\sim}{e}(u,t)$ be

$$\phi(s,t;u) = E[e^{-s\underset{\sim}{e}(u,t)}] \tag{5.1}$$

Now write down a backward differential equation for

$$\psi(s,t;u) = E[e^{-sX(t,u)}], \tag{5.2}$$

$X(t,u)$ being the combined effect at t of all of the (sub, or component) effects occurring after u and before t:

$$X(t,) = \sum_{i=0}^{A(t,u)} \underset{\sim}{e}(u_i,t) = \int_u^t \underset{\sim}{e}(s,t)\, dN(s) \tag{5.3}$$

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where $A(t)$ is the number of (Poisson) events in (u,t), and u_i is the instant at which the ith such event occurs. It is seen that

$$\frac{\partial \psi}{\partial u} = [\lambda(u) - \lambda(u)\ \phi(s,t;u)]\ \psi(s,t;u), \tag{5.4}$$

exactly as was true for (3.6), and thus

$$\psi(s,t;u) = \exp\{\int_u^t \lambda(u)[\phi(\xi,t;v)-1]dv\} \tag{5.5}$$

EXAMPLE. Let $\lambda(u) = \lambda$ and $\underset{\sim}{e}(u,t) = Se^{-\theta(t-u)}$, $\theta > 0$, S having the exponential distribution with density $\mu e^{-\mu x}$. It follows that

$$\phi(s,t;u) = \frac{\mu}{\mu + se^{-\theta(t-u)}} \tag{5.6}$$

and thence that

$$\begin{aligned} \psi(s,t;0) &= \exp\left\{\int_0^t \lambda \frac{se^{-\theta(t-u)}}{\mu + se^{-\theta(t-u)}}\, du\right\} \\ &= \left(\frac{\mu + se^{-\theta t}}{\mu + s}\right)^{\lambda/\theta} \to \left(\frac{\mu}{\mu + s}\right)^{\lambda/\theta} \end{aligned} \tag{5.7}$$

as $t \to \infty$, so in the long run the total effect has gamma distribution. See Gaver and Miller (1962) for the same result derived differently.

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5.1. Zipf's Law and Pareto Tails

Let $\lambda(u) = e^{\alpha u}$ and let $e(u,t) = S(u)\, e^{\theta(t-u)}$, $\theta > 0$, $S(u)$ having the distribution $F(\ )$; successive $S(u)$ values are independent. This setup models a collection of organisms that are born at random times and grow independently and exponentially thereafter. We are interested in the fraction of all those born in $(0,t)$ that exceed size x at time t; we shall see that the fraction exhibits the "Pareto tail" associated with Zipf's law; see Mandelbrodt (1978).

Let $\underset{\sim}{I}(x,u,t)$ denote the indicator function

$$\underset{\sim}{I}(x,u,t) = \begin{cases} 1 & \text{if } \underset{\sim}{e}(u,t) > x \\ 0 & \text{if } \underset{\sim}{e}(u,t) \leq x. \end{cases} \tag{5.8}$$

In the present model define

$$p_x(u,t) \equiv E[I(x,u,t)] = \bar{F}(xe^{-\theta(t-u)}) \tag{5.9}$$

although what follows next does not require the latter explicit form. Now introduce the bivariate generating function

$$g(z_a, z_x, t; u) = E[z_a^{A(t,u)} z_x^{N_x(t,u)}] \tag{5.10}$$

where $A(t,u)$ is the number of arrivals (births) in (u,t), and $N_x(t,u)$ is the number of those organisms born in (u,t) that exceed x in size at time t; we put $A(t,0) = A(t)$, and $N_x(t) = N_x(t,0)$. Then by the backward argument analogous to that producing (2.8),

$$\begin{aligned} &g(z_a, z_x, t; u) \\ &= [z_a z_x\, \lambda(u)du\, p_x(u,t) + z_a\, \lambda(u)du\, (1-p_x(u,t)) + 1 - \lambda(u)du] \\ &\qquad \times g(z_a, z_x, t; u+du) + o(du), \end{aligned} \tag{5.11}$$

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which leads to a differential equation with solution

$$g(z_a, z_x, t; 0) = \exp\{\int_0^t \lambda(u)du[z_a(z_x-1)p_x(u,t) + (z_a-1)]\} \quad (5.12)$$

This shows that $A(t)$ and $N_x(t)$ have a bivariate Poisson distribution; from (5.10) one finds

$$m(t) = E[A(t)] = Var[A(t)] = \int_0^t \lambda(u)du ,$$

$$m_x(t) = E[N_x(t)] = Var[N_x(t)] = \int_0^t \lambda(u)\ p_x(u,t)du \quad (5.13)$$

$$Cov[A(t), N_x(t)] = \int_0^t \lambda(u)\ p_x(u,t)du = m_x(t) .$$

Under many interesting circumstances, a notable instance being the specific model beginning this example, both $E[A(t)]$ and $E[N_x(t)] \to \infty$ as $t \to \infty$.

$$E[A(t)] = \int_0^t e^{\alpha u}du = \frac{1}{\alpha}(e^{\alpha t}-1) \sim \frac{1}{\alpha} e^{\alpha t} ,$$

$$E[N_x(t)] = \int_0^t e^{\alpha u}\ du\ \bar{F}(xe^{-\theta(t-u)})$$

$$= e^{\alpha t} \frac{1}{\theta x^{\alpha/\theta}} \int_{xe^{-\theta t}}^x z^{\alpha/\theta-1}\ \bar{F}(z)dz \quad (5.14)$$

$$\sim \frac{e^{\alpha t}}{\theta x^{\alpha/\theta}} \int_0^x z^{\alpha/\theta-1}\ \bar{F}(z)dz$$

provided the integral exists.

Now define

$$f(x,t) = \frac{E[N_x(t)]}{E[A(t)]} \sim \frac{\alpha}{\theta} \frac{1}{x^{\alpha/\theta}} \int_0^x z^{\alpha/\theta-1} \bar{F}(z)dz \quad , (t \to \infty) \quad (5.15)$$

the long-time average fraction of those organisms born before t and that exceed x in size at t; clearly for large x this fraction exhibits the "Pareto tail" behavior: $x^{-\alpha/\theta}$, provided that the integral exists for large x, as will be assumed.

It will now be shown that with high probability the above law should actually hold for observed data in the following sense. Form the ratio of observable random variables $N_x(t)/A(t)$; this ratio should approximate to $f(x,\infty)$ as $t \to \infty$. To show that this is so, consider, for $\varepsilon > 0$,

$$P\left\{\frac{N_x(t)}{A(t)} \leq f(x,t) + \varepsilon\right\} = P\{N_x(t) - A(t)[f(x,t) + \varepsilon] \leq 0\}. \quad (5.16)$$

For the specific model of this example both $E[A(t)]$ and $E[N_x(t)] \to \infty$ as $t \to \infty$, consequently it can be shown (e.g. by the continuity theorem for characteristic functions) that $(A(t), N_x(t))$ are approximately bivariate normal for large t with parameters given by (5.13). Therefore

$$E[N_x(t) - A(t)[f(x,t) + \varepsilon]] = - E[A(t)]\varepsilon$$

$$\begin{aligned} \mathrm{Var}[N_x(t) - A(t)[f(x,t) + \varepsilon]] \\ &= \mathrm{Var}[N_x(t)] + [f(x,t) + \varepsilon]^2 \, \mathrm{Var}[A(t)] \\ &\qquad - 2[f(x,t) + \varepsilon] \, \mathrm{cov}[N_x(t), A(t)] \\ &= m_x(t) + [f(x,t) + \varepsilon]^2 \, m(t) - 2[f(x,t) + \varepsilon] \, m_x(t) \\ &= m(t) \, \{f(x,t) + [f(x,t) + \varepsilon]^2 - 2[f(x,t) + \varepsilon] \, f(x,t)\} \\ &= m(t) \, \{f(x,t)(1 - f(x,t)) + \varepsilon^2\} \end{aligned} \tag{5.17}$$

Now use the normal approximation to assess the probability (5.16):

$$P\left\{\frac{N_x(t)}{A(t)} \le f(x,t) + \varepsilon\right\} = P\{N_x(t) - A(t)[f(x,t) + \varepsilon] \le 0\}$$

$$\cong \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{m(t)\varepsilon/[m(t)\{f(x,t)(1-f(x,t)+\varepsilon^2\}]^{1/2}} e^{-z^2/2} \, dx \tag{5.18}$$

As $t \to 0$ the fraction $f(x,t)$ approaches the (finite) right side of (5.15) and then, since $m(t) \to \infty$, the integral approaches unity; a similar argument shows that $N_x(t)/A(t) > f(x,t) - \varepsilon$ with probability approaching unity. It follows that the ratio $N_x(t)/A(t)$ is a consistent estimator of $f(x,t)$ at the point x as $t \to \infty$. The statement (5.18) can also be used to supply approximate confidence limits for $f(x,t)$.

An alternative formulation leads to similar asymptotic results. Suppose that the arrival rate is now taken to be $k\lambda(u)$, k being a parameter that will later approach infinity; see Barbour (1974) for an analogous model and analysis. The interpretation is that when k becomes large organism births occur thick and fast--even more so, of course, for later times than earlier when (now) $\lambda(u) = ke^{\alpha u}$ as in our example. Now all analysis goes through as before, and the average fraction function is

$$f_k(x,t) = \frac{E[N_x(t)]}{E[A(t)]} = \frac{1}{x^{\alpha/\theta}} \frac{e^{\alpha t}}{\theta(e^{\alpha t}-1)} \int_{xe^{-\theta t}}^{x} z^{\alpha/\theta-1}\,\bar{F}(z)\,dz\,, \tag{5.19}$$

independent of k, while

$$m(t;k) = E[A(t)] = k\int_0^t \lambda(u)\,du = \frac{k}{\alpha}(e^{\alpha t}-1) = km(t) \tag{5.20}$$

As $k \to \infty$ a central limit theorem argument once again applies (here for every finite t) to show that

$$P\left\{\frac{N_x(t)}{A(t)} \le f_k(x,t) + \varepsilon\right\}$$

$$\cong \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{km(t)\varepsilon/[km(t)\{f_k(x,t)(1-f_k(x,t)+\varepsilon^2\}]^{1/2}} e^{-z^2/2}\,dz \tag{5.21}$$

and the latter probability clearly approaches unity as $k \to \infty$, this

time for every t. The asymptotic normality also allows approximate confidence limits to be placed on $f_k(x,t)$.

We emphasize that the above analysis applies just to any single x-value. Analogous results should be derivable for any finite sequence of x-values, and thence extended by continuity to all real values, obtaining results similar to the Glivenko-Cantelli theorem for ordinary distribution functions. This, and other, generalizations are under development and will be reported in a subsequent paper.

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